2022 IEEE 61st Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

Unknown Input Observer-based Output Regulation for Uncertain Minimum Phase Linear Systems Affected by a Periodic Disturbance

Yang Wang, Yizhou Gong, Gilberto Pin, Fanglai Zhu, Andrea Serrani, Thomas Parisini

2022 IEEE 61st Conference on Decision and Control (CDC) | 978-1-6654-6761-2/22/$31.00 ©2022 IEEE | DOI: 10.1109/CDC51059.2022.9992590

***Abstract*— This paper deals with the problem of disturbance rejection for uncertain LTI SISO systems perturbed by an *unmeasurable* external disturbance under the framework of output regulation. The system is assumed to be minimum phase and internally stable, but the model parameters are completely unknown. In addition, no knowledge of the external disturbance, including frequency, amplitude and phase is required to be known in advance. A novel high-order sliding mode-based Unknown Input Observer(UIO) is developed to stabilize the system and reconstruct the external disturbance. The main feature distinguishing the proposed method from the existing ones is that we do not need to integrate a frequency estimator into the adaptive controller or update the frequency estimation in a hybrid manner. Instead, the disturbance is directly duplicated by the aforementioned unknown input observer. The boundedness of states and asymptotic convergence properties are rigorously proved. Finally, the effectiveness of the proposed technique is illustrated by a numerical example.**

I. Introduction

The problem of tracking desired references while rejecting disturbances in the presence of model uncertainties is generically known as robust output regulation, which has played a central role through the history of control theory [1] and can be found in myriad engineering applications, including active rotor balancing [2], active noise cancellation [3] and active suspensions [4], etc. In practice, the reference signal is usually available, whereas the external disturbance to be rejected is more difficult to obtain, especially facing a time-vary uncertain operating environment. In this context, this work focuses on the more challenging task of disturbance rejection and considers the uncertain LTI SISO systems described by:

$$\begin{aligned}\dot{x}(t) &= A(\mu)x(t) + B(\mu)[u(t) - d(t)], \quad x(0) = x_0 \in \mathcal{X}\\ y(t) &= C(\mu)x(t),\end{aligned} \tag{1}$$

where $\mathcal{X} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ represent the state, the input and the output of plant (1), respectively. The periodic external disturbance $d(t) = \psi\cos(\omega^* t + \phi)$ to be rejected is modeled as the output of the following exosystem

$$\dot{w} = Sw, \quad d = \Gamma w \tag{2}$$

with $w(0) = w_0 \in \mathbb{R}^2$, and

$$S = \begin{pmatrix} 0 & 1 \\ -(\omega^*)^2 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The control objective is then posed as designing the control $u(t)$ such that all signals are bounded in closed-loop and the output $y(t)$ is regulated to zero. The celebrated Internal Model (IM) principle [5] suggests that the information regarding the exosystem is essential to solve the problem. However, considering the realistic scenarios in which the plant model uncertainties are inevitable and the external signals can not be directly measured, the construction of a robust regulator that adaptively embeds the internal model of the exosystem is nontrivial [6]. An appealing study is therefore to establish the minimum priori knowledge on exosystem and plant which is required to design a robust adaptive regulator.

Solutions existed in the literature can be roughly classified depending on whether ω^* is known prior. The known frequency case has been extensively investigated [7]–[9]. Recent results in [10], [11] have shown that an effective switching-based AFC can be designed for a completely unknown stable system as long as the frequency responses of the plant over the periods of interest remain in a known compact set. On the other hand, when ω^* is unknown, there are numerous IM-based adaptive techniques [12]–[15] solving this problem under the assumption that an accurate nominal model of the plant is available. Such a critical assumption is relaxed to a minimum phase uncertain system in [16] given that the plant parameters are restricted in a known bounded region. Later, minimum phase requirement is replaced by the internally stable assumption together with certain information on frequency response of the plant [17]. A backstepping-based method is presented in [18], but the complexity of the algorithm dramatically increases with the relative degree.

Despite the fact that there have been considerable discussions on the robust output regulation problem, we believe that some fundamental issues concerning the relaxation of the assumption seem far from settled in the sense that, the latest progress presented in [11], [17] both involve complicated switching mechanisms, along with the projection operation posed on the update laws. Consequently, the unsatisfactory transient behaviour, the slow adaptation requirement stemming from the stability analysis and the potential erratic

Y. Wang and Y. Gong are with the School of Information Science and Technology, ShanghaiTech University, Shanghai, China. G. Pin is with Electrolux Italia, S.p.A., Italy. F. Zhu is with the College of Electronic and Information Engineering, Tongji University, Shanghai, China. Andrea Serrani is with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus OH – USA.; T. Parisini is with Imperial College London, UK, with University of Trieste, Italy, and also with the KIOS Research and Innovation Center of Excellence, University of Cyprus, CY-1678 Nicosia, Cyprus. This work was supported in part by the Yangfan Program of Shanghai, China, under Grant 21YF1429600 and the National Natural Science Foundation of China under Grant 61973236. Corresponding author: Y.Wang, `wangyang4@shanghaitech.edu.cn`

behaviour brought by the non-stop switching are inherent difficulties that hard to overcome.

In this paper, we tend to address this problem from a new perspective that allows us to resort to a sliding-mode-based Unknown Input Observer (UIO) [19] to reconstruct the exosystem in a finite time manner. The novelties of this work are twofold: i) The prior knowledge of the external disturbance is completely removed without incorporating a separate observer to estimate the frequency ii) No hybrid update law, projection or switching mechanism is needed in our scheme, which implies the convergence rate can be made arbitrarily fast in theory. Moreover, from the simulation results, the robustness and superior transient behaviours of the presented method are both observed, which suggests the applicability of the proposed technique in the practical settings.

Notations: The following notations will be utilized in this paper: $\|\cdot\|$ represents the Euclidean norm of the matrice or vector; For any constant matrix $M \in \mathbb{R}^{m\times n}$, denote $M^+ = \max(M,0)$ and $M^- = \max(-M,0)$. Then obviously, we have $M = M^+ - M^-$ and $|M| = M^+ + M^-$, where $|M|$ stands for a $m \times n$ matrix formed by taking the absolute value of every element of M. For a matrix or vector M $> (\geq, <, \leq) 0$ means that all elements of matrix or vector are $> (\geq, <, \leq) 0$ respectively. In addition, a Metzler matrix is a square matrix in which all the off-diagonal components are nonnegative.

II. Problem Formulation

In this section, we will present a novel formulation of the output regulation problem where the unknown exosystem and states are interpreted as the input signals. This allows us to employ a high-order sliding-mode (HOSM)-based UIO to design a certainty-equivalent feedback controller to stabilize the overall system. Let us start with clarifying the assumptions needed on the plant and exosystem.

Plant (1) is largely uncertain in the sense that the dependence of the matrices $A(\mu), B(\mu)$ and $C(\mu)$ on the unknown parameter vector $\mu \in \mathbb{R}^p$ is unknown but $\mu \in \mathbb{R}^p$ is assumed to range on a given compact set, $\mathcal{P} \subset \mathbb{R}^p$. However, system is assumed to be internally stable, robustly with respect to $\mu \in \mathcal{P}$, which is formally stated as follows:

Assumption II.1. There exist constants $c_1, c_2, c_3 \geq 0$ such that the parameterized family $P_x(\mu) : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ of solutions of the Lyapunov equation $P_x(\mu)A(\mu)+A^\top(\mu)P_x(\mu) = -I_n$ satisfies $c_1 I_n \leq P_x(\mu) \leq c_2 I_n$, $\|P_x(\mu)\| \leq c_3$ for all $\mu \in \mathcal{P}$.

Apart from the internal stability, as a primary result, we also assume the system is a minimum phase plant with relative degree equals to one. Denote the transfer function of plant (1) by

$$W_\mu = k_p \frac{Z(s)}{P(s)} = C(\mu)(sI - A(\mu))^{-1}B(\mu), \tag{3}$$

where $Z(s) = s^m + b_{r+1}s^{m-1} + \cdots + b_{n-1}s + b_n$ and $P(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$ are polynomials of order m and n respectively, from which the relative degree of plant (1) is $r =: n - m$, and k_p stands for the unknown high-frequency gain, which, without loss of generality, assumed to be positive. The extra assumption on plant is given below:

Assumption II.2. Plant (1) is minimum phase with relative-degree-unity, that is $Z(s)$ is Hurwitz and $r = 1$.

The relative degree assumption indeed limits the applicability of the proposed method to certain extent, however, it could be relaxed at the price of additional complexity of the algorithm. However, to illustrate the main idea, we keep above assumption in this paper.

The exosystem model (2) maps the *uncertainty* associated with ψ and ϕ to the unknown initial condition $w_0 \in \mathbb{R}^2$ and the *unknown* frequency is incorporated in the dynamic matrix S. In this work, we only assume the norm-bounds on frequency and the amplitude of $d(t)$ are given[1]:

Assumption II.3. The amplitude ψ and frequency ω^* of satisfy the following inequalities

$$\underline{\omega} \leq \omega^* \leq \overline{\omega}, \quad \psi \leq \overline{\psi} \tag{4}$$

for some *known* positive constants $\underline{\omega}$, $\overline{\omega}$ and $\overline{\psi}$.

Now, the problem addressed in this paper can be formally stated as follows:

Problem II.1. Suppose Assumption II.1-II.3 hold for the uncertain system (1) under the effect of a single sinusoidal disturbance generated by the exosystem (2). Design a dynamic feedback controller in the form of

$$\begin{aligned} \dot{\zeta}_c &= \varphi_c(\zeta_c, y), \quad \zeta_c(0) = \zeta_{c,0} \in \mathbb{R}^m \\ u &= h_c(\zeta_c, y) \end{aligned}$$

such that the trajectories of the closed-loop system originating from all initial conditions $x_0 \in \mathcal{X}$, $\zeta_{c,0} \in \mathbb{R}^m$ are bounded and the output of the plant satisfies $\lim_{t\to\infty} y(t) = 0$ for all $\mu \in \mathcal{P}$.

Similar to [10], [11], we first propose an adaptive feedforward controller(AFC) that is constructed as a copy of the exosystem (2) as follows:

$$\begin{aligned} \dot{\hat{w}} &= S_0\hat{w} + \mathit{\Gamma}^\top u_b, \quad \hat{w}(0) = \hat{w}_0 \in \mathbb{R}^2 \\ u &= \mathit{\Gamma}\hat{w}, \end{aligned} \tag{5}$$

where $u_b : \mathbb{R}_+ \mapsto \mathbb{R}$ is an auxiliary control signal to be determined later and

$$S_0 =: \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}$$

with $\omega_0 > 0$ a constant chosen to verify Assumption II.3. Let $\zeta(t) =: \hat{w}(t) - w(t)$ denote the difference of states between the exosystem (2) and its counterpart (5), we obtain

$$\begin{aligned} \dot{\zeta} &= S_0\zeta + \mathit{\Gamma}^\top(u_b + \delta(t)), \\ u - d &= \mathit{\Gamma}\zeta \end{aligned}$$

with $\delta(t) =: \tilde{\mathit{\Omega}} w_1(t)$, where $\tilde{\mathit{\Omega}} =: (\omega^*)^2 - \omega_0^2$ represents the frequency mismatch and w_1 represents the first state of

[1]This assumption is not conservative in the sense that one can always choose sufficiently small (or large) constants to serve as the boundaries.

exosystem (2). Clearly, $\delta(t)$ is also a bounded sinusoidal signal based on Assumption II.3. Changing the coordinate as $z =: x - \Pi(\mu)\zeta$, the interconnection of the plant (1), the exosystem (2) and AFC controller (5) can be rewritten as follows:

$$\begin{aligned}\dot{z} &= A(\mu)z - \Pi(\mu)\Gamma^\top(u_b + \delta), \quad && z(0) = z_0 \in \mathbb{R}^n \\ \dot{\zeta} &= S_0\zeta + \Gamma^\top(u_b + \delta), && \zeta(0) = \zeta_0 \in \mathbb{R}^2 \\ y &= C(\mu)z + \vartheta_0^\top(\mu)\zeta, && \end{aligned} \tag{6}$$

where $\Pi(\mu)$ is a solution of the Sylvester equation

$$\Pi(\mu)S_0 = A(\mu)\Pi(\mu) + B(\mu)\Gamma \tag{7}$$

and set $C(\mu)\Pi(\mu) =: \vartheta_0^\top(\mu)$. The existence and uniqueness of the $\Pi(\mu)$ is guaranteed by the fact that the spectra of S_0 and $A(\mu)$ are disjoint.

Hereafter, the uncertainty parameter vector μ is ignored for neatness when no confusion is caused.

To facilitate the forthcoming analysis, we introduce a coordinate change as $\xi_1 =: y$ and $\xi_2 =: \dot{y}$, and rewrite System (6) into its normal form as follows

$$\begin{aligned}\dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= -\omega_0^2\xi_1 + k_p(u_b + \delta) + CA^2z + \omega_0^2Cz\end{aligned} \tag{8}$$

where we have taken advantage of the fact that $\vartheta_0^\top = C\Pi$, $CB = k_p$ and the Sylvester equation (7).

Remark II.1. It is worth pointing out that the dimension of ξ-subsystem depends on the relative-degree of system (1). For $n - m \geq 2$, we can construct corresponding ξ-system with order $n - m + 1$ and develop the observer-based controller presented in the following Section accordingly.

To facilitate the controller design, we add and subtract a term $k_{p0}u_b$ to the right hand side of the differential equation of ξ_2, where k_{p0} is a control constant.

Then, we can rewrite the dynamic of $\xi =: \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}^\top$ as

$$\begin{aligned}\dot{\xi} &= S_0\xi + \Gamma^\top k_{p0}u_b + \Gamma^\top\Delta, \\ y &= \bar{C}\xi,\end{aligned} \tag{9}$$

where $\bar{C} = \begin{pmatrix}1 & 0\end{pmatrix}$, Δ can be regarded as the lumped uncertainty in the form of

$$\Delta =: (CA^2 + \omega_0^2C)z - \tilde{k}_pu_b + k_p\delta \tag{10}$$

and $\tilde{k}_p = k_{p0} - k_p$ represents the error of high-frequency gain estimate given by a constant k_{p0}.

Now, concerning the closed-loop system in the form of

$$\begin{aligned}\dot{\xi} &= S_0\xi + \Gamma^\top k_{p0}u_b + \Gamma^\top\Delta, \\ \dot{z} &= Az - \Pi\Gamma^\top(u_b + \delta), \\ y &= \bar{C}\xi,\end{aligned} \tag{11}$$

the original Control Problem II.1 can be recast into an observer-based regulator design problem as:

Problem II.2. Under Assumptions II.1-II.2, design a control law of $u_b(t)$ for System (11) such that the trajectories of the closed-loop system are bounded and the output y of the plant asymptotically converges to zero as time goes to infinity.

III. Controller Design

The aim of this section is that of showing the design of the control law u_b which solves the Problem II.2. Thanks to the controllability of the matrix pair $(S_0, \Gamma^\top)$, we propose the following certainty-equivalent control law:

$$u_b = -K\hat{\xi} - \frac{1}{k_{p0}}\hat{\Delta} \tag{12}$$

in which K is a gain matrix selected so as to ensure $S_0 - k_{p0}\Gamma^\top K$ is Hurwitz. $\hat{\xi}$ and $\hat{\Delta}$ stand for the estimates for ξ and Δ, which will be given by, respectively, a high-order sliding-mode (HOSM)-based observer and a novel input re-constructor [20] described later. According to the so-called separation principle [20], it is well-known that such combined observer-controller output feedback scheme is able to preserve the main features of the controller with high-order sliding-mode observers [21].

Recalling the definition of Δ in (10), it holds

$$\begin{aligned}u_b = &-K\hat{\xi} - \frac{1}{k_{p0}}(CA^2 + \omega_0^2C)z \\ &- (\frac{k_p}{k_{p0}} - 1)u_b - \frac{k_p}{k_{p0}}\delta - \frac{1}{k_{p0}}\tilde{\Delta}\end{aligned}$$

where we denote by $\tilde{\Delta} =: \hat{\Delta} - \Delta$ the estimate error of the lumped uncertainty and $\tilde{\xi} =: \hat{\xi} - \xi$ the estimate error of state. Further, re-organizing the equation above, we can write the input signal as

$$u_b = -\frac{k_{p0}}{k_p}K\xi - \frac{1}{k_p}(CA^2 + \omega_0^2C)z - \delta - \frac{k_{p0}}{k_p}K\tilde{\xi} - \frac{1}{k_p}\tilde{\Delta}. \tag{13}$$

Substituting (13) into the close-loop system (11) gives

$$\begin{aligned}\begin{bmatrix}\dot{z} \\ \dot{\xi}\end{bmatrix} &= \mathcal{M}\begin{bmatrix}z \\ \xi\end{bmatrix} + \mathcal{N}\begin{bmatrix}\tilde{\xi} \\ \tilde{\Delta}\end{bmatrix}, \\ y &= \bar{C}\xi\end{aligned} \tag{14}$$

where

$$\mathcal{M} = \begin{bmatrix}A + \mathrm{T} & \frac{k_{p0}}{k_p}\Pi\Gamma^\top K \\ 0 & F\end{bmatrix}, \ \mathcal{N} = \begin{bmatrix}\frac{k_{p0}}{k_p}\Pi\Gamma^\top K & \frac{1}{k_p}\Pi\Gamma^\top \\ -k_{p0}\Gamma^\top K & -\Gamma^\top\end{bmatrix}$$

and the matrices T and F are given by

$$\mathrm{T} = \frac{1}{k_p}\Pi\Gamma^\top(CA^2 + \omega_0^2C), \quad F = S_0 - k_{p0}\Gamma^\top K$$

respectively. In view of (14), the proposed control law u_b is said to be a stabilizing one as long as the system enjoys an input-to-state stable property. Obviously, given $\mathcal{M}$ an upper triangle matrix, the stability property of the overall system merely depends on the structure of the matrices F and $A+\mathrm{T}$. The Hurwitz property of matrix F is ensured by design, whereas the negative semi-definite property of $A + \mathrm{T}$ will be proved later in section IV. Further, the zeroing of output $y(t)$ would be achieved if the estimate errors $\tilde{\xi}$ and $\tilde{\Delta}$ decay to zero sufficiently fast.

Next, the estimators of the state ξ and the lumped uncertainty Δ will be presented in detail, along with their convergence properties.

A. High-order sliding-mode (HOSM) Observer

Inspired by [21], a novel HOSM observer $\hat{\xi} =: \begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^\top$ is proposed as

$$\begin{aligned}\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \hat{\nu}_1, \\ \dot{\hat{\xi}}_2 &= -\omega_0^2 y + k_{p0} u_b + \hat{\nu}_2\end{aligned} \tag{15}$$

where $\hat{\nu} =: \begin{pmatrix}\hat{\nu}_1 & \hat{\nu}_2\end{pmatrix}^\top$ is generated as follows

$$\hat{\nu}_i = \tau_i \left|\hat{\nu}_{i-1}\right|^{(2-i)/(3-i)} \operatorname{sign}(\hat{\nu}_{i-1}), \quad i = 1, 2 \tag{16}$$

with $\hat{\nu}_0 = y - \hat{\xi}_1$ and $\tau_i > 0$ are sufficiently large tuning gains selected to verify the inequalities:

$$\tau_2 > L \quad \text{and} \quad \frac{2(\tau_2 + L)^2}{\tau_1^2(\tau_2 - L)} < 1 \tag{17}$$

in which $L > 0$ is the Lipschitz constant of the second derivative of ξ_1, i.e. $\|\dot{\xi}_2\| \leq L$. Note that, referring to (8), we do not have the global Lipschitz property for $\dot{\xi}_2$ for now. However, the existence of L for any finite time interval is trivial since the plant is internally stable by Assumption II.1 and the proposed control law u_b is an $\mathcal{L}_{\infty e}$ signal[2].

Now, following the similar manner presented in Levant [21], we can establish the finite time convergence property in the next lemma.

Proposition III.1. (Theorem 5 in [21]) Consider System (9), the estimate $\hat{\xi}$ given by the HOSM observer (15) with (16) is able to track the true state ξ in a finite time $T_f \in \mathbb{R}_+$ if the tuning gains τ_i verify condition (17) for all $t \in [0, T_f]$. Moreover, the convergence time T_f is irrelevant to the Lipschitz constant L but mainly depends on that size of $\|\dot{\hat{\nu}}_0(0)\|$, i.e. the initial estimation error of $\hat{\xi}_2$.

Remark III.1. [23] provides a good pair of choice of the tuning parameters that balances the convergence speed and accuracy is $\tau_1 = 1.5L^{\frac{1}{2}}$and $\tau_2 = 1.1L$, even though they do not satisfy (17). In practice, we can always find proper tuning parameters starting from sufficiently large τ_1 and τ_2.

Further, with the merit of the internal stability in Assumption II.1 and the fact $u_b \in \mathcal{L}_{\infty e}$, we have the signal $\Delta(t)$ defined in (10) belonging to $\mathcal{L}_{\infty e}$ as well.

Lemma III.1. There exists a class-$\mathcal{K}$ function $\beta(t)$ such that, for any $t_f > 0$, $\Delta(t)$ is norm-bounded by $\|\Delta(t)\| \leq \beta(t_f)$ for any $t \in [0, t_f)$.

Proof. Recall the form of $\Delta = (CA^2 + \omega_0^2 C)z - \tilde{k}_p u_b + k_p \delta$. Thanks to Assumption II.1, state $z(t)$ in System (11) features the input-to-state stability with respect to the input signals that consist of a sinusoidal signal δ and a $\mathcal{L}_{\infty e}$ signal u_b. Therefore, $z \in \mathcal{L}_{\infty e}$, which in turn, implies $\Delta(t)$ can be norm-bounded by a class $\mathcal{K}$ function $\beta(t_f)$. □

[2]A signal is said to be a $\mathcal{L}_{\infty e}$ if its $\mathcal{L}_\infty$-norm exists for any finite time t [22]. The boundedness property for the truncated u_b will be more obvious after we determine the estimator for Δ later in Subsection III-B.

Finally, subtracting the first equation in (11) from (15), the dynamics of the estimate error can be obtained as

$$\begin{aligned}\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \hat{\nu}_1, \\ \dot{\tilde{\xi}}_2 &= \hat{\nu}_2 - \Delta\end{aligned} \tag{18}$$

where $\begin{pmatrix}\tilde{\xi}_1 & \tilde{\xi}_2\end{pmatrix}^\top = \tilde{\xi}$. Similar to Lemma III.1, it is shown in [21] that a sliding mode appears on the manifold $\tilde{\xi}_1 = \tilde{\xi}_2 = 0$ in a finite time by choosing the gains $\tau_i > 0$ properly. In addition, this also suggests the estimation error is a bounded and monotonically decreasing signal (See Theorem 5 in [21] for details).

B. Interval Observer-based Estimator

In the remaining part of this section, we will develop an interval-observer-based estimator for Δ (10) that also features a finite-time convergence property. To this end, a non-conservative assumption needs to be made first for the output y and its time derivative as follows:

Assumption III.1. There exist two known constants $\overline{\xi}(0)$ and $\underline{\xi}(0)$ conforming to $\underline{\xi}(0) \leq \xi(0) \leq \overline{\xi}(0)$ for all $x_0 \in \mathcal{X}$ and $\mu \in \mathcal{P}$.

The properties in lemmas listed below will play a critical role in designing an interval observer for ξ-system (9) and consequently an estimator for Δ :

Lemma III.2. [24] The solution of a differential equation $\dot{x}(t) = \mathcal{A}x(t) + d_x(t)$ satisfies $x(t) \geq 0$ for all $t \geq 0$ if $x(0) \geq 0$, the matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a *Metzler* and *Hurwitz* matrix, besides $d_x(t) \in \mathbb{R}^n$, $d_x(t) \geq 0$, for any $t \geq 0$.

Lemma III.3. [24] Suppose there exist column vector variables $\overline{x}(t)$, $x(t)$, $\underline{x}(t) \in \mathbb{R}^n$ satisfying $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for all $t \geq 0$, then for any constant matrix $\mathcal{E} \in \mathbb{R}^{m\times n}$, we have

$$\mathcal{E}^+\underline{x}(t) - \mathcal{E}^-\overline{x}(t) \leq \mathcal{E}x(t) \leq \mathcal{E}^+\overline{x}(t) - \mathcal{E}^-\underline{x}(t).$$

The proof of above lemmas III.2 and III.3 can be found in [25], [26] and therefore omitted here. Consider a coordinate change $\varsigma = Q\xi$ with Q being an invertible matrix, then the ξ-system (9) is transformed into

$$\begin{aligned}\dot{\varsigma} &= QS_0Q^{-1}\varsigma + Q\Gamma^\top k_{p0}u_b + Q\Gamma^\top \Delta, \\ y &= \bar{C}Q^{-1}x.\end{aligned} \tag{19}$$

Thanks to the observability of the matrix pair $\begin{pmatrix}S_0 & \bar{C}\end{pmatrix}^\top$, an interval observer for System (19) is constructed as:

$$\begin{aligned}\dot{\overline{\varsigma}} &= QS_0Q^{-1}\overline{\varsigma} + Q\Gamma^\top k_{p0}u_b + Q\gamma(y - \bar{C}Q^{-1}\overline{\varsigma}) \\ &\quad + (Q\Gamma^\top)^+\beta_1 - (Q\Gamma^\top)^-(-\beta_1), \\ \dot{\underline{\varsigma}} &= QS_0Q^{-1}\underline{\varsigma} + Q\Gamma^\top k_{p0}u_b + Q\gamma(y - \bar{C}Q^{-1}\underline{\varsigma}) \\ &\quad + (Q\Gamma^\top)^+(-\beta_1) - (Q\Gamma^\top)^-\beta_1,\end{aligned} \tag{20}$$

in which $\beta_1 =: \beta(t_f)$ is determined by Lemma III.1 and the initial states are set as $\underline{\varsigma}(0) = Q^+\underline{\xi}(0) - Q^-\overline{\xi}(0)$ and $\overline{\varsigma}(0) = Q^+\overline{\xi}(0) - Q^-\underline{\xi}(0)$. System (20) is the so-called interval observer whose property is asserted by the following Proposition.

Proposition III.2. Under Assumption III.1, the states of System (19) and (20) verify $\underline{\varsigma}(t) \leq \varsigma(t) \leq \overline{\varsigma}(t)$ for all $t \geq 0$ if the gain matrix γ together with matrix Q is chosen such that matrix $Q(S_0 - \gamma\bar{C})Q^{-1}$ is not only *Hurwitz* but also *Metzler*[3] .

The proof can be found in [20, Theorem 2]. Then, thanks to $\underline{\varsigma}(t) \leq \varsigma(t) \leq \overline{\varsigma}(t)$ in Proposition III.2 and Lemma III.3 with the fact $\xi(t) = Q^{-1}\varsigma(t)$, the upper and lower boundary estimates of $\xi(t)$ can be calculated by

$$\begin{aligned}\overline{\xi}(t) &= (Q^{-1})^{+}\overline{\varsigma}(t) - (Q^{-1})^{-}\underline{\varsigma}(t),\\ \underline{\xi}(t) &= (Q^{-1})^{+}\underline{\varsigma}(t) - (Q^{-1})^{-}\overline{\varsigma}(t).\end{aligned} \tag{21}$$

Now, based on the sliding mode observer (15) and the interval observer produced by (20)-(21), we are ready to employ an algebraic unknown input reconstruction method proposed by Zhu [20] to estimate the lumped uncertainty Δ in (10). In virtue of (21), it is not difficult to check that $\underline{\xi}(t) \leq \xi(t) \leq \overline{\xi}(t)$ holds for all $t \geq 0$, which suggests that $\underline{\xi}_2 \leq \xi_2 \leq \overline{\xi}_2$, then there must exist a time varying $\alpha_2(t)$ satisfying $0 \leq \alpha_2(t) \leq 1$ such that $\xi_2 = \alpha_2\overline{\xi}_2 + (1-\alpha_2)\underline{\xi}_2$ or

$$\xi_2 = \alpha_i(t)(\overline{\xi}_2 - \underline{\xi}_2) + \underline{\xi}_2. \tag{22}$$

Then differentiating (22) gives

$$\dot{\xi}_2 = \dot{\alpha}_2(\overline{\xi}_2 - \underline{\xi}_2) + \alpha_2(\dot{\overline{\xi}}_2 - \dot{\underline{\xi}}_2) + \dot{\underline{\xi}}_2. \tag{23}$$

In view of (21), it follows that

$$\begin{aligned}\overline{\xi} - \underline{\xi} &= \left[(Q^{-1})^{+} + (Q^{-1})^{-}\right]\overline{\varsigma} - \left[(Q^{-1})^{-} + (Q^{-1})^{+}\right]\underline{\varsigma}\\ &= \left|Q^{-1}\right|\widehat{\varsigma}\end{aligned} \tag{24}$$

with $\widehat{\varsigma} =: \overline{\varsigma} - \underline{\varsigma}$ whose dynamic is governed by

$$\dot{\widehat{\varsigma}} = Q(S_0 - \gamma\bar{C})Q^{-1}\widehat{\varsigma} + \left|Q\Gamma^{\top}\right|\widehat{\beta}_1 \tag{25}$$

where $\widehat{\beta}_1 =: 2\beta_1$. Further, referring to (20) and (21), it holds that

$$\begin{aligned}\dot{\underline{\xi}} &= (Q^{-1})^{+}\dot{\underline{\varsigma}} - (Q^{-1})^{-}\dot{\overline{\xi}}\\ &= M_1\underline{\varsigma} - M_2\overline{\varsigma} + \gamma y + N_1(-\beta_1) - N_2\beta_1 + \Gamma^{\top}k_{p0}u_b\end{aligned}$$

where

$$\begin{aligned}M_1 &= (Q^{-1})^{+}Q(S_0 - \gamma\bar{C})Q^{-1},\\ M_2 &= (Q^{-1})^{-}Q(S_0 - \gamma\bar{C})Q^{-1},\\ N_1 &= (Q^{-1})^{+}(Q\Gamma^{\top})^{+} + (Q^{-1})^{-}(Q\Gamma^{\top})^{-},\\ N_2 &= (Q^{-1})^{+}(Q\Gamma^{\top})^{-} + (Q^{-1})^{-}(Q\Gamma^{\top})^{+}.\end{aligned}$$

As a result, in terms of $\dot{\xi}_2$, above equations suggest that

$$\begin{aligned}\overline{\xi}_2 - \underline{\xi}_2 &= f_1(\widehat{\varsigma}),\\ \dot{\overline{\xi}}_2 - \dot{\underline{\xi}}_2 &= f_2(\widehat{\varsigma}),\\ \dot{\underline{\xi}}_2 &= f_3(\overline{\varsigma}, \underline{\varsigma}) + k_{p0}u_b\end{aligned} \tag{26}$$

[3]Readers are referred to [27], [28] for systematic and concrete design procedures like solving Sylvester Equation or through time-varying coordinates to derive Q, γ.

where

$$f_1(\widehat{\varsigma}) = \Gamma\left|Q^{-1}\right|\widehat{\varsigma},$$

$$f_2(\widehat{\varsigma}) = \Gamma\left|Q^{-1}\right|\left[Q\left(S_0 - \gamma\bar{C}\right)Q^{-1}\widehat{\varsigma} + \left|Q\Gamma^{\top}\right|\widehat{\beta}_1\right],$$

$$f_3(\overline{\varsigma}, \underline{\varsigma}) = \Gamma(M_1\underline{\varsigma} - M_2\overline{\varsigma} + \Gamma^{\top}k_{p0}u_b + \gamma y + N_1(-\beta_1) - N_2\beta_1).$$

Now, substituting (26) into (23) yields

$$\dot{\xi}_2 = \dot{\alpha}_2 f_1(\widehat{\varsigma}) + \alpha_2 f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) + k_{p0}u_b. \tag{27}$$

Meanwhile, recall the first equation of (11), we have $\dot{\xi}_2 = -\omega_0^2 y + k_{p0}u_b + \Delta$ which, together with (27), gives

$$\Delta = \dot{\alpha}_2 f_1(\widehat{\varsigma}) + \alpha_2 f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) + \omega_0^2 y \tag{28}$$

from which (28), a re-constructor for the unknown input Δ in (11) is obtained by

$$\hat{\Delta} = \hat{\dot{\alpha}}_2 f_1(\widehat{\varsigma}) + \hat{\alpha}_2 f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) + \omega_0^2 y \tag{29}$$

where $\hat{\dot{\alpha}}_2$ and $\hat{\alpha}_2$ are the estimation of $\dot{\alpha}_2$ and α_2, respectively. Then, due to (22), $\hat{\alpha}_2$ can be computed by

$$\hat{\alpha}_2 = \frac{\hat{\xi}_2 - \underline{\xi}_2 - \epsilon}{\overline{\xi}_2 - \underline{\xi}_2 + \epsilon} \tag{30}$$

with $\epsilon = 1$, if $\overline{\xi}_2 = \underline{\xi}_2$; otherwise, $\epsilon = 0$.

In order to get the estimation of $\dot{\alpha}_2$ denoted by $\hat{\dot{\alpha}}_2$, we again resort to a class of second-order high-gain sliding model observer [21] as follows:

$$\begin{aligned}&\dot{\rho}_1 = \iota_1, \iota_1 = -\kappa_1|\rho_1 - \hat{\alpha}_2|^{1/2}\text{sign}(\rho_1 - \hat{\alpha}_2) + \rho_2,\\ &\dot{\rho}_2 = -\kappa_2\text{sign}(\rho_2 - \iota_1)\end{aligned} \tag{31}$$

where ρ_2 is the identical estimate of $\dot{\alpha}_2$, used as the exact estimate of $\dot{\alpha}_2$ as well mentioned in the design of HOSM observer (15), as long as two positive scalar gains $\kappa_i > 0, i = 1, 2$ are determined properly based on Proposition III.1.

Proposition III.3. Under Assumption III.1, the estimator for the lumped uncertainty Δ in (10) that consists of (29)-(31) and (20) is able to provide an accurate estimation within finite time, that is, there exists a time instant $\bar{T}_f > 0$ such that $\tilde{\Delta}(t) = 0$, for all $t \geq \bar{T}_f$.

Proof. From (28) and (29), we can deduce that

$$\tilde{\Delta}(t) = \tilde{\dot{\alpha}}_2(t)f_1(\widehat{\varsigma}) + \tilde{\alpha}_2(t)f_2(\widehat{\varsigma})$$

where $\tilde{\dot{\alpha}}_2(t) =: \hat{\dot{\alpha}}_2(t) - \dot{\alpha}_2(t)$ and $\tilde{\alpha}_2(t) =: \hat{\alpha}_2 - \alpha_2$. Thanks to Lemma III.1, the finite time convergence of $\tilde{\xi}_2(t)$ implies $\tilde{\alpha}_2(t) = 0$ for all $t \geq T_f$. Since (31) features the same structure with (15), one can easily conclude that, after another period of time, say T_{ff}, $\hat{\dot{\alpha}}_2$ must converge to $\dot{\alpha}_2$ and the local boundedness of $\hat{\dot{\alpha}}_2$ is certain with all signals bounded. Then, setting $\bar{T}_f =: T_f + T_{ff}$, we have $\tilde{\Delta}(t) = 0$ for all $t \geq \bar{T}_f$. Thus complete the proof. □

Remark III.2. The finite time convergence properties stated in Lemma III.1 and Proposition III.3 also indicate the boundedness of the estimate error $\tilde{\xi}$ and $\tilde{\Delta}$. Referring to the close-loop System (14), we can conclude that the boundedness of ξ

and z is guaranteed if (14) is stable in the sense of Lyapunov (will be rigorous proved in Theorem IV.1), which in turn, suggests the boundedness of $\hat{\xi}$ and $\hat{\Delta}$. Also, In view of (8) and (13), the boundedness of $\|\dot{\xi}_2\|$ now becomes trivial. As a result, the Lipschitz condition required by (17) actually holds globally.

IV. Stability Analysis

The previous section presented a delicate design of a certainty-equivalence controller (12) utilizing a HOSM-based observer (15) and an interval state-based estimator (29) that both enjoy the finite time convergence properties as long as u_b remains bounded in that time interval. Now, a natural question arise is that what happens after the estimation errors reach zero? What if the control signal u_b increases to infinity later? Referring to (13) and thanks to the fact that the estimation errors $\tilde{\xi}$ and $\tilde{\Delta}$ are norm-bounded and will quickly decay to zero, one can conclude that the boundedness of u_b for all $t \in \mathcal{R}_+$ would highly depend on the boundedness of ξ and z. This can be shown via the stability analysis of the closed-loop system (14), which is the main task of this section.

First, a lemma that plays a key role in establishing the stability property is given below:

Lemma IV.1. Suppose Assumption II.2 holds, then matrix $A+\mathrm{T} := \Omega$ defined in (14) is a semi-definite negative matrix.

Proof. Recall the definition of the matrix

$$\mathrm{T} = \frac{1}{k_p}\Pi\Gamma^\top(CA^2+\omega_0^2 C),$$

in which Π is a solution of the Sylvester equation (7). Now, split Π matrix into two parts as $\Pi = [h_1, h_2]$, where h_1 and h_2 are both a n-dimensional column vector. Then, the Sylvester equation (7) can be rewritten into two equations as follows:

$$Ah_1 = -w_0^2 h_2, \quad Ah_2 + B = h_1. \tag{32}$$

Eliminating the vector h_1 in (32) and putting the terms involving h_2 together, we obtain

$$(A^2 + w_0^2 I_n)h_2 = -AB,$$

which yields

$$h_2 = -(A^2 + w_0^2 I_n)^{-1}AB \tag{33}$$

due to the nonsingularity of $A^2 + w_0^2 I_n$. Now, bearing in mind that $\Pi\Gamma^\top = h_2$, we can substitute (33) into Ω to get

$$\begin{aligned}\Omega &= A - \frac{1}{k_p}(A^2+w_0^2 I_n)^{-1}ABC\left(A^2+w_0^2 I_n\right)\\ &= (A^2+w_0^2 I_n)^{-1}\left(A - \frac{1}{k_p}ABC\right)(A^2+w_0^2 I_n)\end{aligned}$$

which suggests the eigenvalues of Ω are irrelevant to ω_0, but purely determined by the structure of the plant, i.e. A, B, C and k_p. Hence, we now focus on the structure property of the matrix $A - \frac{1}{k_p}ABC =: \bar{\Omega}$. Notice that, since the proposed algorithm does not use any specific knowledge of the parameter value of the plant, without loss of generality, we can always assume the matrices A, B, C are already in a controllable canonical form as follows:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & \cdots & \cdots & -a_1 \end{bmatrix}_{n\times n}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_p \end{bmatrix}_{n\times 1},$$
$$C = \begin{bmatrix} b_n & \cdots & b_2 & 1 \end{bmatrix}_{1\times n}$$

where $\left[a_{1,\cdots,}a_n\right]^\top \subset \mathbb{R}^n$ and $\left[b_{2,\cdots,}b_n\right]^\top \subset \mathbb{R}^{n-1}$ are the coefficients of the transfer function defined in (3). Certainly, these are unknown coefficients that depend on the unknown parameter vector $\mu \in \mathcal{P}$.

Given a controllable form, after some tedious calculations, one can observe that $\bar{\Omega}$ admits the form of

$$\bar{\Omega} = \begin{bmatrix} \mathcal{B} & \mathbf{0_{n-1\times 1}} \\ \mathcal{A}_{1\times(n-1)} & 0 \end{bmatrix}_{n\times n} \tag{34}$$

where $\{\mathcal{A}\}_{1,i} = a_1 b_{n+1-i} - a_{n+1-i}$, $i = 1, 2, \cdots n-1$ and

$$\mathcal{B} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_n & -b_{n-1} & \cdots & -b_2 \end{bmatrix}_{(n-1)\times(n-1)}.$$

It is trivial to see the eigenvalues of $\mathcal{B}$ happen to be the zeros of plant. Thanks to the minimum phase assumption, the matrix $\bar{\Omega}$ therefore has $n-1$ negative poles plus one at the origin, which indicates matrix Ω is semi-definite negative. Thus end the proof. □

Finally, the main results of our stability analysis are summarized by the following theorem.

Theorem IV.1. Consider the closed-loop system consisting of the plant (14), the control law (12), the observer (15) and the estimator (29), suppose Assumptions II.1-II.2 hold, then all trajectories initiating from $x \in \mathcal{X}$, $\hat{\Delta}(0) \in \mathbb{R}$ and $\hat{\xi}(0) \in \mathbb{R}^2$ verifying Assumption III.1 are uniformly bounded and the output y converges to zero asymptotically.

Proof. We first conduct a coordinate change to state z of the closed-loop system (14) that $\bar{z}(t) =: \mathcal{H}z(t)$ with $\mathcal{H} = A^2 + \omega_0^2 I_n$ being an invertible matrix and verify the condition $\mathcal{H}\Omega\mathcal{H}^{-1} = \bar{\Omega}$ introduced in Lemma IV.1. Referring to (14), we have the dynamic of $\bar{z}$ is given by

$$\dot{\bar{z}} = \bar{\Omega}\bar{z} + \mathcal{H}\mathcal{N}\begin{pmatrix}\tilde{\xi} \\ \tilde{\Delta}\end{pmatrix}.$$

Thanks to the block structure of $\bar{\Omega}$ shown in (34), we can split the state $\bar{z}$ into two signals as $\bar{z} = \begin{pmatrix}\bar{z}_1 & \bar{z}_2\end{pmatrix}^\top$ where $\bar{z}_1 \in \mathbb{R}^{n-1}$ and $\bar{z}_2$ is a scalar. Accordingly, the dynamics of $\bar{z}$ can be rewritten as:

$$\begin{aligned}\dot{\bar{z}}_1 &= \mathcal{B}\bar{z}_1 + \mathcal{I}_1\xi + \mathcal{I}_1\tilde{\xi} + \mathcal{O}_1\tilde{\Delta},\\ \dot{\bar{z}}_2 &= \mathcal{A}\bar{z}_1 + \mathcal{I}_2\xi + \mathcal{I}_2\tilde{\xi} + \mathcal{O}_2\tilde{\Delta}\end{aligned} \tag{35}$$

where the matrices are partitioned as

$$\frac{k_{p0}}{k_p}\mathcal{H}\Pi\Gamma^\top K = \begin{bmatrix}\mathcal{I}_{1(n-1)\times 2}\\ \mathcal{I}_{2(1\times 2)}\end{bmatrix}, \ \frac{1}{k_p}\mathcal{H}\Pi\Gamma^\top = \begin{bmatrix}\mathcal{O}_{1(n-1)\times 1}\\ \mathcal{O}_{2(1\times 1)}\end{bmatrix}$$

and $\mathcal{A}$, $\mathcal{B}$ are defined in (34).

Due to the Hurwitz property of matrices $\mathcal{B}$ and F, one can easily conclude that the subsystem consisting of $\bar{z}_1$ and ξ is input-to-state stable with respect to the estimate errors $\tilde{\xi}$ and $\tilde{\Delta}$. Moreover, the unforced $\bar{z}_1$-ξ-subsystem is actually exponentially stable. Thus, thanks to the finite convergence and boundedness properties claimed in Proposition III.1 and III.3, we have signals $\bar{z}_1$, ξ and y are all $\mathcal{L}_\infty$ signals and will exponentially decay to zero after $t \geq \bar{T}_f$. Nevertheless, to conclude the uniform boundedness of all trajectories, one thing left now is to show that $\bar{z}_2$ also belongs to $\mathcal{L}_\infty$.

According to Theorem 5.1 in [29], we have, for each $\bar{z}_1(0) \in \mathbb{R}^{n-1}$, $\xi(0) \in \mathbb{R}^2$, the $\bar{z}_1$-ξ-subsystem is finite-gain $\mathcal{L}_p$ stable for any $p \in [1,\infty]$, that is, there exist nonnegative constants $\varkappa_1$, $\varkappa_2$, ϱ_1 and ϱ_2 such that

$$\begin{aligned}\|\bar{z}_p\|_{\mathcal{L}_{pe}} &\leq \varkappa_1\left(\|\tilde{\xi}\|_{\mathcal{L}_{pe}} + \|\tilde{\Delta}\|_{\mathcal{L}_{pe}}\right) + \varrho_1,\\ \|\xi\|_{\mathcal{L}_{pe}} &\leq \varkappa_2\left(\|\tilde{\xi}\|_{\mathcal{L}_{pe}} + \|\tilde{\Delta}\|_{\mathcal{L}_{pe}}\right) + \varrho_2. \end{aligned} \tag{36}$$

In our case, together with the finite convergence of $\|\tilde{\xi}\|$ and $\|\tilde{\Delta}\|$, it follows that $\xi \in \mathcal{L}_1$ and $\bar{z}_1 \in \mathcal{L}_1$. Integrating both sides of the differential equation for $\bar{z}_2$ given in (35) yields

$$\begin{aligned}\|\bar{z}_2(t)\| \leq \|\bar{z}_2(0)\| + \int_0^t \varpi_2\|\xi(\tau)\| + \varpi_1\|\bar{z}_1(\tau)\|d\tau\\ + \int_0^{\bar{T}_f}(\varpi_3\|\tilde{\xi}(\tau)\| + \varpi_4\|\tilde{\Delta}(\tau)\|)d\tau \end{aligned} \tag{37}$$

for some positive constants ϖ_1, ϖ_2, ϖ_3, ϖ_4. To be further, we have

$$\begin{aligned}\|\bar{z}_2(t)\|_{L_\infty} = \sup_{t\geq 0}\|\bar{z}_2(t)\| \leq \|\bar{z}_2(0)\| + \varpi_2\|\xi\|_{L_1}\\ + \varpi_1\|\bar{z}_1\|_{L_1} + \varpi_3\|\tilde{\xi}\|_{L_1} + \varpi_4\|\tilde{\Delta}\|_{L_1} < \infty. \end{aligned} \tag{38}$$

Hence $\bar{z}_2(t) \in \mathcal{L}_\infty$, which means all signals are bounded. Thus end the proof. □

V. Numerical Example

In this section, we show the effectiveness of the proposed scheme by a numerical example[4]. Consider a second-order minimum phase system described by

$$G(s) = \frac{2s+1}{s^2+2s+3}$$

under the effect of a sinusoidal external disturbance

$$d(t) = \begin{cases} 2\sin(2t - \frac{\pi}{3}) & \text{if } t \in [0,15) \\ 2\sin(3t - \frac{\pi}{3}) & \text{if } t \geq 15 \end{cases}$$

The reduplicated exosystem is constructed with $\omega_0 = 1$ and $\hat{\omega}(0) = [0,0]^\top$. The feedback gain matrix K used in the control law (12) is selected such as the eigenvalues of $S_0 - k_{p0}\Gamma^\top K$ are -5 and -5.2 with $k_{p0} = 1$. The tuning gains for the HOSM observer (15) and (31) are selected as: $\tau_1 = 40, \tau_2 = 30$ and $\kappa_1 = 15, \kappa_2 = 12$. As there are no precise rules for the selection of the HOSM observer gains in (15) and (31), this was done via trial and error, starting from the sufficiently large value, see [21] and [20] for some discussions in this respect. The parameters of the interval observer (20) are selected as $\gamma = [17, 51]^\top$ and the transformation matrix used in (20) $Q = \begin{pmatrix} 0.2353 & -0.0588 \\ 0.1529 & -0.0118 \end{pmatrix}$ is chosen so that

$$Q(S_0 - \gamma\bar{C})Q^{-1} = \begin{pmatrix} -4 & 0 \\ 0 & -13 \end{pmatrix}$$

is a Hurwitz and Metzler matrix verifying the condition in Proposition III.2. The initial condition of the plant is set as: $x(0) = [-1,1]^\top$, while the HOSM observers in (15) and (31) are initialized with : $\hat{\xi}(0) = [-1,0]^\top$, $\rho_1(0) = 0.5$, $\rho_2(0) = 0$. Thus, according to Assumption III.1, the initial value of interval observer (20) can be calculated as $\overline{\xi} = [1,3]^\top$ and $\underline{\xi} = [-1,1]^\top$.

Simulation results are shown in Fig. 1-3. With the fast convergence of control input u to unknown disturbance d in Fig. 1, the output of the plant is regulated very quickly to zero. The finite-time convergence properties of HOSM observers in Proposition III.1 and III.3 are verified by the Fig. 2. As depicted in Fig. 3, the boundedness of all signals is ensured. The dramatic features enjoyed by the proposed method are demonstrated via the transient behaviour around 15s, at which the frequency of external disturbance undergoes an abrupt change. From the magnified plot in Fig. 1, y goes back to zero after a small oscillation while the input u tracks the new disturbance instantaneously. Another distinguished feature of the proposed scheme is that we do not estimate the frequency of the external disturbance and change the structure of the controller accordingly. With $\omega_0 = 1$, the complete rejection of the periodic signal with different harmonic tunes is achieved. Certainly, after the output being regulated to zero, one can obtain the frequency information applying many existing parameter estimation techniques to the control signal $u(t)$.

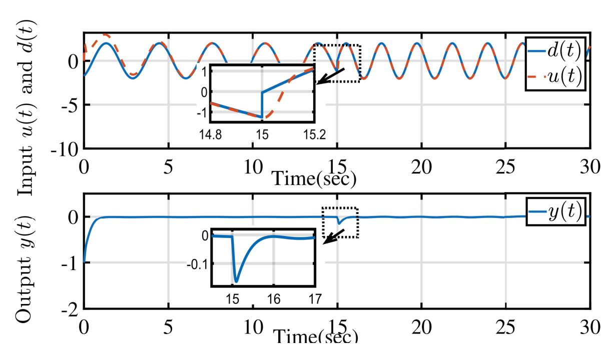

Fig. 1. Time history of the output and input signals.

[4]Due to space limitation, further comparison simulation with the IM-adaptive method [30] are not presented here, which emphasises the fast convergence ability of the proposed method. The interested reader can refer to the supplementary materials via the link https://www.overleaf.com/read/pbbrswqmsfyg

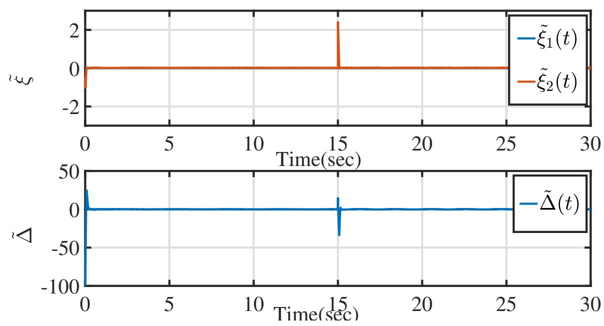

Fig. 2. The estimate errors of ξ and Δ.

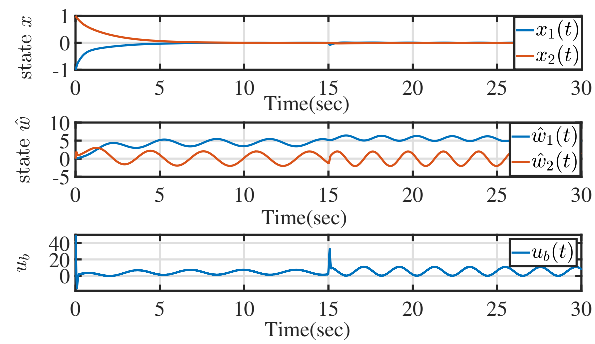

Fig. 3. Time history of states of plant (1) and controller (12).

VI. Conclusions

In this paper, a novel UIO-based AFC is proposed to solve the output regulation problem for an uncertain LTI SISO system with few information on the nominal model and external disturbance. Compared with the switching adaptive mechanism [11], [31], we relax the requirement for frequency information ω^* without introducing any discontinuity in the controller. We have shown the resulting closed-loop system enjoys asymptotically stability and the disturbance can be completely rejected. The simulation results are consistent with the theoretical results and show that the control objective is achieved by the proposed scheme. The future investigation should include the removal of minimum phase requirement and extend the algorithm to the case in which the system has higher relative degree.

References

[1] H. W. Knobloch, D. Flockerzi, and A. Isidori, *Topics in Control Theory*. Basel: Birkhäuser Basel, 1993.

[2] S. Zhou and J. Shi, “Active balancing and vibration control of rotating machinery: a survey,” *Shock and Vibration Digest*, vol. 33, no. 5, pp. 361–371, 2001.

[3] C. Hansen, S. Snyder, X. Qiu, L. Brooks, and D. Moreau, *Active Control of Noise and Vibration, Second Edition*. Crc Pr I Llc, 2012.

[4] I. D. Landau, A. Constantinescu, and D. Rey, “Adaptive narrow band disturbance rejection applied to an active suspension - An internal model principle approach,” *Automatica*, vol. 41, no. 4, pp. 563–574, 2005.

[5] B. Francis and W. M. Wonham, “The Internal Model Principle of Control Theory,” *Automatica*, vol. 1098, no. August, pp. 457–465, 1976.

[6] R. Marino and P. Tomei, “Adaptive output regulation for minimum-phase systems with unknown relative degree,” *Automatica*, vol. 130, p. 109670, 2021.

[7] M. Kamaldar and J. B. Hoagg, “Adaptive Harmonic Control for Rejection of Sinusoidal Disturbances Acting on an Unknown System,” *IEEE Transactions on Control Systems Technology*, vol. 28, no. 2, pp. 277–290, mar 2020.

[8] J. B. Hoagg, M. A. Santillo, and D. S. Bernstein, “Internal Model Control in the Shift and Delta Domains,” *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1066–1072, May 2008, conference Name: IEEE Transactions on Automatic Control.

[9] S. Pigg and M. Bodson, “Adaptive algorithms for the rejection of sinusoidal disturbances acting on unknown plants,” *IEEE Transactions on Control Systems Technology*, vol. 18, no. 4, pp. 822–836, 2010.

[10] Y. Wang, G. Pin, A. Serrani, and T. Parisini, “Switching-based sinusoidal disturbance rejection for uncertain stable linear systems,” in *2018 Annual American Control Conference (ACC)*, 2018, pp. 4502–4507.

[11] Y. Wang, G. Pin, A. Serrani, and T. Parisini, “Removing SPR-like conditions in adaptive feedforward control of uncertain systems,” *IEEE Transactions on Automatic Control*, vol. 65, no. 6, pp. 2309–2324, 2020.

[12] A. Serrani, “Rejection of harmonic disturbances at the controller input via hybrid adaptive external models,” *Automatica*, vol. 42, no. 11, pp. 1977–1985, 2006.

[13] R. Marino and P. Tomei, “Output regulation for linear minimum phase systems with unknown order exosystem,” *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 2000–2005, 2007.

[14] G. Fedele and A. Ferrise, “Biased sinusoidal disturbance compensation with unknown frequency,” *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3207–3212, 2013.

[15] S. Jafari and P. A. Ioannou, “Robust adaptive attenuation of unknown periodic disturbances in uncertain multi-input multi-output systems,” *Automatica*, vol. 70, pp. 32–42, 2016.

[16] R. Marino and P. Tomei, “An adaptive learning regulator for uncertain minimum phase systems with undermodeled unknown exosystems,” *Automatica*, vol. 47, no. 4, pp. 739–747, 2011.

[17] R. Marino and P. Tomei, “Hybrid Adaptive Multi-Sinusoidal Disturbance Cancellation,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4023–4030, aug 2017.

[18] C. T. Yilmaz and H. I. Basturk, “Output feedback control for unknown lti systems driven by unknown periodic disturbances,” *Automatica*, vol. 99, pp. 112–119, 2019.

[19] F. Zhu, “State estimation and unknown input reconstruction via both reduced-order and high-order sliding mode observers,” *Journal of Process Control*, vol. 22, no. 1, pp. 296–302, 2012.

[20] L. Ma, F. Zhu, J. Zhang, and X. Zhao, “Leader-Follower Asymptotic Consensus Control of Disturbance Reconstruction Approach,” *IEEE Transactions On Cybernetics*, pp. 1–12, 2021.

[21] A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *International Journal of Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[22] P. A. I. J. Sun, *Robust Adaptive Control*. Prentice Hall, 1995.

[23] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, *Sliding mode control and observation*, 1st ed. Birkhäuser Basel, 2014.

[24] Z. Zhou, F. Zhu, D. Xu, and Z. Gao, “An interval-estimation-based anti-disturbance sliding mode control strategy for rigid satellite with prescribed performance,” *ISA Transactions*, vol. 105, pp. 63–76, 2020.

[25] D. Efimov, L. Fridman, T. Raïssi, A. Zolghadri, and R. Seydou, “Interval estimation for LPV systems applying high order sliding mode techniques,” *Automatica*, vol. 48, no. 9, pp. 2365–2371, 2012.

[26] L. Farina and S. Rinaldi, *Positive linear systems: theory and applications*. John Wiley & Sons, 2000, vol. 50.

[27] F. Mazenc and O. Bernard, “Interval observers for linear time-invariant systems with disturbances,” *Automatica*, vol. 47, no. 1, pp. 140–147, 2011.

[28] TarekRaïssi, D. Efimov, and A. Zolghadri, “Interval State Estimation for a Class of Nonlinear Systems,” *IEEE Transactions On Automatic Control*, vol. 23, no. 2, pp. 383–394, 2013.

[29] H. K.KHALIL, *Nonlinear Systems*. Prentice Hall, 1996.

[30] R. Marino and P. Tomei, “An adaptive learning regulator for uncertain minimum phase systems with undermodeled unknown exosystems,” *Automatica*, vol. 47, no. 4, pp. 739–747, 2011.

[31] Y. Wang, G. Pin, A. Serrani, and T. Parisini, “Switching-based rejection of multi-sinusoidal disturbance in uncertain stable linear systems under measurement noise,” in *Proceedings of the IEEE Conference on Decision and Control*, 2019, pp. 6112–6117.